NOTES AND QUERIES.

[Under this head we propose to give brief notices of whatever is remarkable as connected with the theory, practice, or history of Assurance here or abroad, and to afford an opportunity for inquiries in relation thereto.]

THE following mode of deducing the formula for an endowment assurance will perhaps be new to some of our readers.

The sum assured is to be paid at a given age, if that age be attained; or at death, should it happen previously. But this would exactly be the case with an assurance calculated from a table of mortality terminating at the age in question. So that we have only to treat the annuity obtained from such a table in the same way that the annuity for the whole term of life is used, in order to arrive at the usual results. Hence the annual premium for an endowment assurance will be $1 \div \frac{N_{x-1} - N_{x+n-1}}{D_x} - d.$

Or thus:—

The single premium for a temporary assurance is $\frac{1 + A^{t-1}}{1 + \rho} - A^t$, (*see* Baily, vol. i., p. 149,) and for the endowment $\frac{a}{a}(1 + \rho)^{-n}$.

But $\frac{1 + A^{t-1}}{1 + \rho} - A^t + \frac{a}{a}(1 + \rho)^{-n} = \frac{1 - \rho A^{t-1}}{1 + \rho}$; the usual expression for the single premium, A^{t-1}, being substituted for A.

Or, reasoning as in Milne, (vol. i., p. 167,)

$$\frac{1}{1 - v} : \frac{v}{1 - v} - A^{t-1} :: 1 : v - (1 - v)A^{t-1}.$$

The following ingenious method of approximating to the value of ρ, in the equation $A = \frac{(1 + \rho)^n - 1}{\rho}$, we have received from Mr. Edward Ryley, the able actuary of the Australasian Assurance Company:—

The *amount* of an annuity being represented by A, the *rate of interest* by ρ, and the *number of years* by n, we have

$$A = \frac{(1 + \rho)^n - 1}{\rho} \ldots . (a),$$

as the value of A in terms of ρ and n.

It is required to eliminate ρ from this equation.

[In Baily's "Interest and Annuities," p. 123, are collected the methods employed by Simpson, Halley, and the author himself. Mr. Baily's formula is very accurate, but practically useless on account of its complexity.]

From equation (a) we have

$$A\rho + 1 = (1 + \rho)^n,$$

and

$$\log. (A\rho + 1) = n \log. (1 + \rho) \ldots . (b).$$

Expanding each side of this equation, we have

$$2M \left\{ \frac{A\rho}{A\rho + 2} + \frac{1}{3}\left(\frac{A\rho}{A\rho + 2}\right)^3 + \frac{1}{5}\left(\frac{A\rho}{A\rho + 2}\right)^5 + \&c. \right\}$$

$$= 2nM \left\{ \frac{\rho}{\rho + 2} + \frac{1}{3}\left(\frac{\rho}{\rho + 2}\right)^3 + \frac{1}{5}\left(\frac{\rho}{\rho + 2}\right)^5 + \&c. \right\} \ldots \ldots (b),$$

where M is the modulus of the system of logarithms employed.

These series have the advantages of being generally convergent, and, the terms being always fractional, of proceeding by the uneven powers of the terms.

But the two sides are very unequally convergent: for $\frac{A\rho}{A\rho + 2}$, being always fractional, approaches without limit to unity as A increases; whilst $\frac{\rho}{\rho + 2}$ is always in practice so small a fraction, that the uneven powers of it, after the first, which enter into the series, may be neglected without sensible error.

Neglecting all the terms except the first, on both sides, we obtain

$$\rho = \frac{2(A - n)}{(n - 1)A} \ldots . (c).$$

This is a very near approximation when the rate of interest is low and n small.

Thus, at 3 per cent., for $n = 3$, $n = 5$, and $n = 10$, the answers are $\rho = \cdot 0294$, $\rho = \cdot 0291$, and $\rho = \cdot 0284$.

For $n = 50$, the answer is $\rho = \cdot 0227$.

The errors then are

$$\begin{aligned} \text{for } n &= 3;\ \cdot 0006 \\ &= 5;\ \cdot 0009 \\ &= 10;\ \cdot 0016 \\ &= 50;\ \cdot 0073, \text{ or nearly 1 per cent.} \end{aligned}$$

But this approximate value of ρ being thus obtained, we may by the use of the same method of development obtain a much nearer value.

Call the value of ρ thus obtained ρ', and put $\rho' + \rho'' = \rho$ in the equation (b); which thus becomes

$$\log. (A\overline{\rho' + \rho''} + 1) = n \log. (1 + \rho' + \rho''),$$

$$\text{or } \log. (A\overline{\rho' + 1} + A\rho'') = n \log. (1 + \rho') \left(1 + \frac{\rho''}{1 + \rho'}\right);$$

which developed becomes

$$\log. (A\rho' +) 1 + 2M \left\{ \frac{A\rho''}{2(A\rho' + 1) + A\rho''} + \&c. \&c.\right.$$

$$= n \log. (1 + \rho') + 2nM \left\{ \frac{\rho''}{2(1 + \rho') + \rho''} + \&c. \&c. \right.$$

In this expression, it is evident that if ρ' be a moderately near approximation to ρ, the terms $A\rho''$ and ρ'', which occur in the denominators on either side, are very small in comparison to $2(A\rho' + 1)$ and $2(\rho' + 1)$, to which they are added. Omitting these terms, $A\rho''$ and ρ'', from the denominators, and putting $A\rho' + 1 = K$ and $\rho' + 1 = R$, we obtain

$$\rho'' = \frac{RK (\log. K - n \log. R)}{M (nK - AR)};$$

or, substituting, except in the factor $\log. K - n . \log. R$, the values of R and K, we obtain

$$\rho'' = \frac{\left\{ 1 + (A\rho' + A + 1) \rho' \right\} (\log. K - n \log. R)}{M \left\{ n + (\overline{n-1} . \rho' - 1) A \right\}}.$$

If in this expression we omit the term multiplied by ρ'^2 from the numerator, we shall obtain finally

$$\rho'' = \frac{\{1 + (1 + A) \rho'\} (\log. K - n . \log. R)}{M \left(n + (\overline{n-1} . \rho' - 1) A\right)} \ldots . (d).$$

If we correct the values of ρ', previously found (p. 333), by this formula, we obtain

for $n = 3$; $\rho'' = \cdot 0009$
„ $n = 5$; $\rho'' = \cdot 0009$
„ $n = 10$; $\rho'' = \cdot 0016$
„ $n = 50$; $\rho'' = \cdot 0084$

Applying these corrections to the values of ρ', at p. 333, we have

for $n = 3$; $\rho = \cdot 0303$
„ $n = 5$; $\rho = \cdot 0300$
„ $n = 10$; $\rho = \cdot 0300$
„ $n = 50$; $\rho = \cdot 0311$

and the final errors are

for $n = 3$; $\cdot 0003$
„ $n = 5$; $\cdot 0000$
„ $n = 10$; $\cdot 0000$
„ $n = 50$; $\cdot 0011$

If we apply the same methods to $n = 10$, $A = 13 \cdot 1808$, we shall find as the result of equation (c), $\rho' = \cdot 0536$, in which the error is $\cdot 0064$. The correction by equation (d) gives $\rho'' = \cdot 0068$; so that $\rho = \rho' + \rho'' = \cdot 0604$, being a final error of $\cdot 0004$.

NOTE.—The equation (d) is more readily calculated in the form

$$\rho'' = \frac{(A\rho' + \rho' + 1) (\log. K - n \log. R)}{M\{(n-1) A\rho' - (A - n)\}}.$$

Mr. Augustus De Morgan, Professor of Mathematics in University College, and Hon. Sec. of the Astronomical Society, has favoured us with the following note "On the equivalence of Compound Interest with Simple Interest paid when due."

At one time there was much discussion upon the value of an annuity at simple interest, and the apparent paradox that a perpetual annuity at simple interest is of infinite value. The reader may find this so-called difficulty discussed in the *Supplement* of the *Penny Cyclopædia*, article "Rebate:" it arises from the suppositions made in the rule of discount *at simple interest* being different from those on which the annuity is supposed to be paid. It seems now to be tolerably well agreed that £20 will pay a perpetual annuity of £1, at simple interest, merely because the interest is always paid away in satisfaction of the claim before any interest upon interest can be made. And yet there can be no doubt that if the buyer of the annuity divides his £20 into $v+v^2+v^3+\ldots$, where $v=1\div 1{\cdot}05$, and chooses to believe that v, v^2, &c. are paid at the end of the first, second, &c. years, he receives those sums with compound interest. Accordingly, it is admitted that a perpetual annuity has the same value whether money makes simple or compound interest. But I am not aware that it has ever been asserted that the value of a *terminable annuity* is the same thing whether simple or compound interest be supposed. Simple interest, paid when due, *is* compound interest, if it be granted that the actual holder of money may make interest. The fiction of the old rules lies in this, that the actual holder is to make no interest of any money which was ever received under the name of interest. Nevertheless, I have never met with any work in which the equivalence of simple interest, paid when due, and of compound interest, is actually shown.

To show it in the case of a terminable annuity, proceed as follows. A person lends £1 to be repaid by £a at the end of each of n following years. On the theory of compound interest, the loan of £1 is divided into av, av^2. av^3, &c., and these portions are repaid with compound interest at the end of one, two, three, &c. years. On the theory of simple interest, k standing for $\{(+r)^n-1\}\div r$, the loan of £1 is divided into the following portions:—

$$\frac{1}{k},\quad \frac{1+r}{k},\quad \frac{(1+r)^2}{k},\ldots\ldots\ \frac{(1+r)^{n-1}}{k},$$

and at the end of each year, one of these portions is paid as instalment of principal, together with interest upon all which remained due at the end of the last year. This in every case, makes up £a, the payment made under the compound interest scheme.

To show this, let V_m represent the sum which remains due at the end of the mth year, where £a is made to pay simple interest and instalment. Then at the end of the $(m+1)$th year, rV_m of interest is due, and $a-rV_m$ is left to reduce the principal. Hence

$$V_{m+1}=V_m-(a-rV_m);\ \text{or,}\ V_{m+1}=(1+r)\,V_m-a.$$

The solution of this equation of differences, upon the supposition that we begin with $V_0=1$, is

$$V^m = \frac{a}{r} - \left(\frac{a}{r} - 1\right)(1+r)^m;$$

and if we require that the principal shall be extinguished in n years, or $V_n = o$, we must have

$$a = \frac{(1+r)^m \cdot r}{(1+r)^n - 1}; \text{ or, } \frac{r}{1-v^n},$$

the same as in the compound interest scheme. Further, the instalment paid at the end of the mth year, $V_{m-1} - V_m$, will be found to be $(a-r)(1+r)^{m-1}$, as above asserted, for $(a-r)k = 1$.

Suppose the following question;—There is a debt of £b, to pay which there is a yearly fund of £a, which is first to pay interest, and then to go in reduction of principal; what debt remains at the end of m years? If we let the debt remain entire, at compound interest, it will in m years be $b(1+r)^m$: and if we make a sinking fund, also at compound interest, of the redeeming annuity, we shall in m years have $a\left\{(+r)^m - 1\right\} \div r$. If, then, simple interest paid when due, and compound interest, be of identical effect, the debt remaining at the end of m years will be

$$b(1+r)^m - a\,\frac{(1+r)^m - 1}{r};$$

or,

$$\frac{a}{r} - \left(\frac{a}{r} - b\right)(1+r)^m,$$

which is precisely what we should get from such a process as we used above, making $V_o = b$ instead of $V_o = 1$.